Spin and Statistics

There is a fundamental connection between the spin of a particle and the symmetry of the many particle wave functions. Bose had proposed Bose statistics for photons which translates to symmetric wave functions for many-photon states, and Fermi and Dirac had shown that the Pauli principle could be translated into the antisymmetric wave functions for many spin 1/2 particles. When quantum mechanics and subsequently quantum field theory were developed the question received more attention. The history of ideas up to and including Pauli's work is systematically presented in the monograph "Pauli and the Spin-Statistics Theorem" [1]. His proof depended on invoking relativistic invariance; but the major impact of particle statistics is in the non-relativistic domain like atomic structure, nuclear structure, phonons in a solid, theory of metals, and theory of superconductivity.

Schwinger's version of Quantum Theory is derived from the Least Action Principle. Sudarshan used this to prove the Spin-Statistics theorem for non-relativistic particles [2, 3]. For this proof the following conditions are the only ones necessary:

1. the Lagrangian is invariant under $SU(2)$ transformations;
2. the Lagrangian must be expressed in a basis in terms of hermitian fields $\xi = \xi^\dagger$;
3. the kinematic term is linear in the time derivatives of the fields;
4. the kinematic term has a bilinear dependence on the fields ξ.

The Schwinger Lagrangian is a very general one that satisfies all these conditions:

$$\mathcal{L} = \frac{i}{2} K^0_{rs} \left(\xi_r \dot{\xi}_s - \dot{\xi}_r \xi_s \right) - \frac{i}{2} K^j_{rs} \left(\xi_r \cdot \nabla_j \xi_s - \nabla_j \xi_r \cdot \xi_s \right) + \xi_r M_{rs} \xi_s \equiv \xi_r \Lambda_{rs} \xi_s \tag{1}$$

where

$$\Lambda = \frac{i}{2} K^0 \overleftrightarrow{\partial}_t - \frac{i}{2} K^j \overleftrightarrow{\partial}_j + M \tag{2}$$

with

$$\overleftrightarrow{\partial} = \overrightarrow{\partial} - \overleftarrow{\partial} \tag{3}$$

Sudarshan showed that this connection can be obtained for the symmetry of the bilinear scalars in the field expressed in terms of hermitian components. Due to the properties of

$SO(3)$ invariance, or its covering group $SU(2)$, the scalar product for integral spin is:

$$(U, V) = \sum_{r,s} U_r V_s \delta_{rs}, \tag{4}$$

where U and V multiply like vectors to give a scalar product. The scalar product for half-integer spin is:

$$(\psi, \phi) = \sum_{r,s} \psi_r \phi_s \left(i\sigma_y\right)_{rs}, \tag{5}$$

which implies that they multiply like spinors.

Under a rotation, the kinematic term in the Lagrangian should transform as:

$$\xi_r \Lambda_{rs} \xi_s \to \xi_r \Lambda_{rs} \xi_s = +\xi_r \Lambda_{rs} \xi_s \tag{6}$$

for Bose statistics, while

$$\xi_r \Lambda_{rs} \xi_s \to \xi_r \Lambda_{rs} \xi_s = -\xi_r \Lambda_{rs} \xi_s \tag{7}$$

for Fermi statistics. This product is symmetric for integer spin and antisymmetric for half integer spin. Using only these conditions, appropiate K^0 and K^j matrices can be chosen so that the equations of motion reduce to the non-relativistic wave equations for integer and half-integer spin respectively. Sudarshan presented these results in his 1968 paper [3] and subsequent publications including a monograph [1], and in a paper which deals with fields with internal symmetries [4].

It should be noted that the spatial derivative and mass terms need not be bilinear for this result to hold [5].

That the symmetry of the many particle wave functions could correspond to Bose, Fermi, or more general statistics was shown to depend on the connectivity of the configuration space [6]. In a series of papers, Imbo and Sudarshan concluded that the connectivity of the space does not impose a restriction on the statistics [7]. Attempts to establish such a connection were shown to "beg the question".

When the theory is applied to 2-space dimensions we get the Anyon statistics, discovered by Wilczek [8]. The statistics can be generalized provided only the equations of motion which are trilinear in the field components are used, as shown by Green [9]. Ryan and Sudarshan showed that all representation of para fermions are spinors in suitable Euclidean spaces [10]. Mehta, Mukunda, Sharma, and Sudarshan have studied coherent states for para Bose fields [11].

Generalized statistics (q-oscillators and f-oscillators) have been studied by Man'ko, Marmo, Sudarshan, and Zaccaria [12]. Comparison with experimental data has been carried out by other groups to determine the upper limit on the f- and q-parameters.

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