Quantum Optical Coherence: Sudarshan Representation

Quantum theory had its origin in the study of blackbody radiation. The quantized electromagnetic field had Bose particles as quanta; yet we used classical wave optics to deal with the phenomena of reflection, refraction, polarization, interference, and diffraction. Light sources can interfere only if they are coherent, where coherence is measured by the correlation function of the amplitudes. Wolf showed that these correlation functions propagate as waves. Interference depends upon the bilocal coherence function. For processes which are nonlinear functions like intensity correlations or photoelectric counts, other many-fold quantities are involved. From the beginning of the last century, the notion of partial coherence was introduced into wave optics about the same time as Michelson constructed the Stellar Interferometer.

When photoelectric counting statistics was studied, it involved higher order coherence functions. The same was true also for the Hanbury-Brown and Twiss intensity correlation. For a fixed intensity, the mean count was proportional to the intensity but the number of actual counts was distributed according to the Poisson distribution. When natural light was employed, the distribution was no longer Poisson, but Bose:

$$p(n) = (1 - x)x^n. \tag{1}$$

This could be understood if natural light was an intensity ensemble obtained from Gaussian (bivariate) distribution of the real and imaginary parts of the amplitude. This formula also gave the correct result for the intensity correlation found by Hanbury-Brown and Twiss.

Yet the photoelectric effect is primarily a quantum process and *its quantum nature should manifest itself*. It was known for quite some time that since the electric and magnetic fields do not commute, they cannot simultaneously vanish. In particular, the 'vacuum' - the state of no photons, did not correspond to zero fields. The mean square uncertainty is given by Heisenberg's relation. Instead of using the real fields, we can use the complex combination which acting on the vacuum annihilates it. Already in 1927, Schrödinger showed that the generic minimum uncertainty states result for a class of states which are obtained by shifting the annihilation operator by a *complex* constant. This can be done for every *mode* of the electromagnetic field. It appeared from counting distributions that a tuned laser beam is in a such a coherent state.

The early work on coherent wave optics had an implicit distribution of coherent states with a classical nonnegative density. A statistical state of the quantized electromagnetic field need not have such a nonnegative density distribution; the more so since the state of a fixed number of photons cannot be represented with only a nonnegative density distribution.

In 1963, Sudarshan gave the first and only formulation for *any* state in terms of a quantum distribution function which may take on negative values [1, 2]. The density matrix of a quantum optical field is given by

$$\rho = \int \phi(\{z\}) \left|\{z\}\right\rangle \left\langle\{z\}\right| d^2\{z\}, \tag{2}$$

where $|\{z\}\rangle$ denotes coherent states for all modes of the field. The distribution function $\phi(\{z\})$ becomes negative for any *fixed number* of photons in *any* mode and is now singular. The distribution $\phi(\{z\})$ for a single mode is the solution of an integral equation [3]

$$\phi(z) = \int \langle\zeta|\rho|\zeta\rangle \exp(\zeta^*\zeta) \cdot \exp(-z^*\zeta + z\zeta^*) d^2\zeta. \tag{3}$$

The possibility of a negative distribution can be immediately recognized from this formula. For a finite number of photons in this mode, it is the Fourier transform of

$$\frac{1}{N}(\zeta^*\zeta)^N \exp(\frac{1}{2}\zeta^*\zeta). \tag{4}$$

This indefinite nature of the distribution, when it obtains, has startling *physical* consequences: there can be sub-Poissonian photocounts. The variance of the counts for a Poisson process is

$$\langle n^2\rangle - \langle n\rangle^2 = \langle n\rangle\,. \tag{5}$$

For a probabilistic distribution of the parameters, the Poisson distribution

$$p(n) = \int \omega(\mu) p(n;\mu) d\mu, \tag{6}$$

$\langle n^2\rangle - \langle n\rangle^2$ is always greater than the mean [4]. If the distribution over the Poisson parameter is nonpositive definite - as befits a truly quantum mechanical state - there may be sub-Poisson statistics. This is photon anti-bunching [5]. For a standard (nonnegative) probability distribution of amplitudes, photons tend to be 'bunched' together. But in quantum optics there are states where the photon counts 'anti-bunch'. We can have intensity anti-correlation, i.e., negative Hanbury-Brown and Twiss effect. This is clearly a sign of the nonpositivity of the distribution function $\phi(\{z\})$. This too has been experimentally verified.

An interesting class of quantum optical states is furnished by the so called squeezed states in which the electric and magnetic fields are scaled inversely. Kimble (and his students) have obtained the best squeezed states exceeding 60% squeezing [6]. In such cases the function $\phi(\{z\})$ becomes the Fourier transform of a field which is anti-Gaussian, i.e., increasing as $\exp(\zeta^* + \zeta)$. These clearly lead to nonpositive highly singular quantum distributions. This further confirms the nonpositive nature of the quantum distribution given by Sudarshan.

The theory first discovered by Sudarshan is thus a truly quantum theory of the electromagnetic field which applies to all states of the radiation.

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