Quantum Mechanics of Open Systems and Stochastic Maps

The evolution of a closed quantum state $\rho(t)$ can be represented in the form:

$$\rho(t) = U(t, t_0)\rho(t_0)U^{\dagger}(t, t_0). \tag{1}$$

The time dependence is completely described by the unitary matrix $U(t, t_0)$, or equivalently, by the hermitian Hamiltonian matrix $H(t)$. This type of evolution is natural, but it is not the most general nor adequate for many applications such as relaxation phenomena and irreversibility.

Quantum systems can rarely be thought of as being in total isolation. A system which is interacting with its surroundings is called an open system. Boltzmann's collision formula is a well known example; as is the classical Newton's Law of Cooling. A quantum mechanical example would be a metastable quantum state that can undergo decay.

In experiments it is not possible to completely isolate a quantum system from its environment. For a quantum system, a level of probabilistic description arises in the specification of the density matrix. The equation of motion for the density matrix may be also stochastic, rather than Hamiltonian. This would be the case if the system is 'open' with outside influences on its evolution. These systems deal with temporal changes that are not unitary and cannot be evolved with a Hamiltonian scheme.

For a bipartite system $\rho^A \otimes \rho^B$ governed by a Hamiltonian $H(t)$, one can calculate the reduced evolution of one part:

$$\rho^A(t) = \mathrm{Tr}_B \left[U(t, t_0)\rho^A \otimes \rho^B U^{\dagger}(t, t_0) \right]. \tag{2}$$

The state dynamics could be described by a linear non-Hamiltonian process. More generally, the evolution of a state represented by the density matrix ρ_{rs} is given by the stochastic map B of the form:

$$\rho'_{r's'} = B_{rr',ss'}\rho_{rs}. \tag{3}$$

The general formalism of this is the method of stochastic maps developed by Sudarshan

[1]. The properties of the super-matrix B were found to be:

$$Trace: \qquad B_{rr',rs'} = \delta_{r's'}, \tag{4}$$

$$Hermiticity: \qquad B_{rr',ss'} = B^*_{ss',rr'}, \tag{5}$$

$$Positivity: \qquad z^*_{rr'} B_{rr',ss'} z_{ss'} \geq 0 \tag{6}$$

The theory of completely positive dynamical maps has a lot of similarities with Markovian evolution and stochastic processes. The stochastic map can be written in terms of its eigenmatrices:

$$B_{rr',ss'} = \sum_{\alpha} \eta_{rr'}(\alpha)\eta^*_{ss'}(\alpha) \tag{7}$$

where its positivity is evident [2]. The generally accepted view was that all such maps which could lead to physically meaningful evolutions had to be completely positive. However, the stochastic dynamics of quantum systems may be obtained by the partial trace of a system coupled to another undergoing unitary transformations. The maps thus obtained are not "completely positive". This would be physically acceptable for a restricted class of density matrices. Using a Schrödinger picture, Jordan, Shaji, and Sudarshan have worked out the allowed dynamics and exhibited them explicitly [3]. There are negative maps that have a physical interpretation, such as the ones that come from the reduction of an initially entangled system. Not-completely positive maps are now associated with decoherence control techniques.

These maps do not form a group. However, if a semigroup is considered, a differential form analogous to Boltzmann's equation can be found [4, 5].

Duck, Stevenson and Sudarshan [6] gave a fresh analysis of the provocative paper of Aharanov et al, titled "Can the measurement of the spin of the electron give a value 100?" [7]. They also suggested an experiment to measure the weak birefringence in a crystal; the experiment was carried out by Hulet at Rice University and the prediction was verified [8].

The Pancharatnam-Berry phase which a quantum state acquires while undergoing a cyclic evolution can be used to fine-tune the frequency of a laser. Kimble, Simon, and Sudarshan showed that repeated traversals of a crystal which is rotating about the axis parallel to the laser beam does indeed shift the frequency of light [9].

With the rise of quantum computation, the terrain was found ripe for the applications of Sudarshan's Stochastic Maps. Quantum information is fragile under the influence of the environment. These maps are constantly used to characterize the environment, model decoherence, and correct for undesirable effects. Also, these maps have been useful for studying the separability of bipartite systems, which is of much interest in quantum information where entanglement is considered a resource. For the past decade, Sudarshan and his students have been studying various aspects of this. They have studied the detailed evolution of bipartite systems and the methods of modelling and controlling decoherence [10].

Whenever a measurement is performed on a quantum system, it is essential to have a classical 'meter'. The interaction between the quantum system and the meter is part of the dynamics of the open system. The von Neumann postulates *do not* provide a mechanism for the 'measurement'. Sudarshan has constructed a general theory [11] of measurement which accomplishes this. He has applied it to the Stern-Gerlach experiment [12–14].

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